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INTERSECTION PATTERNS OF CONVEX SETS

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ABSTRACT
Let $K_1, \dots, K_n$ be convex sets in R^d. For $0 \leqq i < n$ denote by f_i the number of subsets S of $\{1, 2, \dots, n\}$ of cardinality $i+1$ that satisfy $\cap \{K_i : i \in S\} \neq \emptyset$. We prove:

THEOREM. *If $f_{d+r} = 0$ for some $r \geqq 0$, then*

$$f_{k-1} \leqq \sum_{i=0}^{d} \binom{r}{k-i}\binom{n-r}{i} \qquad \textit{for all } k > 0.$$

This inequality was conjectured by Katchalski and Perles. Equality holds, e.g., if $K_1 = \cdots = K_r = R^d$ and $K_{r+1}, \dots, K_n$ are $n-r$ hyperplanes in general position in R^d. The proof uses multilinear techniques (exterior algebra). Applications to convexity and to extremal set theory are given.

1. Introduction

The main purpose of this paper is to answer the following question, raised by M. Katchalski and M. A. Perles:

Let $\mathcal{K} = \{K_1, K_2, \dots, K_n\}$ be a family of n convex sets in R^d such that each $(d+r+1)$-subfamily has empty intersection. What is the maximum possible number of $(d+1)$-subfamilies (more generally, $(d+k)$-subfamilies) of $\mathcal{K}$ with non-empty intersection?

Katchalski and Perles conjectured that this maximum (for any k) is attained when $K_1 = K_2 = \cdots = K_r = R^d$, and $K_{r+1}, \dots, K_n$ are hyperplanes in general position; so, for example, the maximum for $k = 1$ is $\binom{n}{d+1} - \binom{n-r}{d+1}$. For $r = 1$ the conjecture was proved independently by M. A. Perles and P. Frankl. We prove this conjecture in section 3. The method of proof is based on tools from exterior algebra. The necessary algebraic background is presented in section 2. The same method turned out to be useful in solving some other combinatorial problems,

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see [5, 6]. As a corollary of the Katchalski–Perles conjecture we obtain in section 4 a sharp version of a theorem of Katchalski and Liu [7], which says roughly the following: If $\mathcal{K}$ is a family of n convex sets in R^d, and if *most* of its $\binom{n}{d+1}$ (d + 1)-subfamilies have non-empty intersection, then $\mathcal{K}$ has an intersecting subfamily that contains *most* of the members of $\mathcal{K}$. Other related results are also obtained. The proof of the Katchalski–Perles conjecture depends on the fact discovered by Wegner [14] that the nerve of a family of convex sets in R^d is d-collapsible. In section 5 we observe that the inequalities obtained in section 3 hold even if the assumption of d-collapsibility is somewhat relaxed, and this leads to some interesting results in extremal finite set theory. The final section 6 contains some remarks and open questions.

The Katchalski–Perles conjecture was independently proved by J. Eckhoff (to appear). We will discuss Eckhoff's proof in section 6.

2. Algebraic background

The material presented here is self-contained and seems sufficient for our purposes. The reader interested in a fuller treatment of exterior algebra may consult [2] or [12].

Let $V = E^n$ be the n-dimensional euclidean space, with the fixed standard orthonormal basis $(e_1, \ldots, e_n)$. Put $N = \{1, 2, \ldots, n\}$; the exterior algebra $\wedge V$ is a 2^n-dimensional inner product space with orthonormal basis $\{e_S : S \subset N\}$. A multilinear associative multiplication $\wedge$ on $\wedge$ V with identity $1 = e_\varnothing$ is defined by the following rules: (a) if $S = \{i_1, \ldots, i_s) \subset N$, $i_1 < i_2 < \cdots < i_s$, then $e_S = e_{i_1} \wedge \cdots \wedge e_{i_s}$ (here e_i is an abbreviation for $e_{\{i\}}$). (b) $e_i \wedge e_j = -e_j \wedge e_i$ for all $i, j \in N$. For $0 \leqq k \leqq n$, $\wedge^k V$ is the subspace of $\wedge V$ spanned by $\{e_S : S \in N^{(k)}\}$. (Here, and elsewhere, $Q^{(k)}$ stands for $\{S \subset Q : |S| = k\}$.) If $(f_1, \ldots, f_n)$ is another basis of V and $S = \{i_1, \ldots, i_s\} \subset N$, $i_1 < \cdots < i_s$, write $f_S = f_{i_1} \wedge \cdots \wedge f_{i_s}$. It is easy to show that $\mathcal{F} = \{f_s : S \in N^{(k)}\}$ is a basis of $\wedge^k V$. Let $A = (a_{ij})_{1 \leqq i,j \leqq n}$ be the transition matrix from the standard basis $(e_i, \ldots, e_n)$ to $(f_1, \ldots, f_n)$, i.e., $f_i = \Sigma\{a_{ij} e_j : j \in N\}$ for $i \in N$.

For S, $T \subset N$ put $A_{S|T} = (a_{ij})_{i \in S, j \in T}$. The transition from the basis $\{e_S : S \in N^{(k)}\}$ of $\wedge^k V$ to $\{f_S : S \in N^{(k)}\}$ is given by

$$f_S = \sum \{(\det A_{S|T}) e_T : T \in N^{(k)}\} \qquad \text{for } S \in N^{(k)}. \tag{2.1}$$

Put $M = \{1, 2, \ldots, m\}$. For an arbitrary $m \times n$ matrix A, and for $1 \leqq k \leqq \min(m, n)$, denote by $C_k(A)$ the matrix $(\det A_{S|T})_{S \in M^{(k)}, T \in N^{(k)}}$. From the discussion above it follows that if A is square and non-singular, then $C_k(A)$ is non-singular as well. Therefore, if the columns of a rectangular matrix A are linearly independent, then so are the columns of $C_k(A)$.

Let $(f_1, \dots, f_n)$ be an orthonormal basis of V. It follows from the Cauchy–Binet Theorem that the basis $\{f_S : S \in N^{(k)}\}$ of $\wedge^k V$ is also orthonormal.

For $g, f \in \wedge V$ the *left interior product* $g \llcorner f \in \wedge V$ is defined by the requirement: $\langle u, g \llcorner f \rangle = \langle u \wedge g, f \rangle$ for all $u \in \wedge V$. If $f \in \wedge^{k+d} V$, $g \in \wedge^d V$ then $g \llcorner f \in \wedge^k V$. It follows from the definition that $h \llcorner (g \llcorner f) = (h \wedge g) \llcorner f$. If $f_1, \dots, f_n$ is any orthornormal basis of V, then

$$f_T \llcorner f_S = \begin{cases} \pm f_{S \setminus T} & \text{if } S \supset T, \\ 0 & \text{if } S \not\supset T. \end{cases} \tag{2.2}$$

We shall deal *throughout the whole paper* with a fixed orthonormal basis $f_1, \dots, f_n$ of V, having the property that *every* square submatrix of the transition matrix A from $(e_1, \dots, e_n)$ to $(f_1, \dots, f_n)$ is non-singular.

The existence of such a matrix A can be established by repeated application of the following procedure: Suppose B is an $n \times n$ orthogonal matrix, S, $T \subset N$, $|S| = |T| = k$, and $\det B_{S|T} = 0$. There must be another set $T^* \in N^{(k)}$, such that $\det B_{S|T^*} \neq 0$. Therefore we can find two sets T', $T'' \in N^{(k)}$ such that $(T' \lhd T'') = 2$, and $\det B_{S|T'} = 0 \neq \det_{S|T''}$. Assume that $1 \in T' \setminus T''$, $2 \in T'' \setminus T'$. Multiply B on the right by the following orthgonal matrix:

$$\begin{pmatrix} \cos\varepsilon & \sin\varepsilon & & & & 0 \\ -\sin\varepsilon & \cos\varepsilon & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & 0 & & & \cdot & \\ & & & & \cdot & \\ & & & & & 1 \end{pmatrix}$$

and call the resulting matrix C. If $\varepsilon > 0$ is sufficiently small, then for all P, $Q \subset N$ with $|P| = |Q|$, $\det B_{P|Q} \neq 0 \Rightarrow \det C_{P|Q} \neq 0$, and, in addition, $\det C_{S|T'} \neq 0$.

3. The main theorem

We start with a few definitions: Let $\mathcal{K} = \{K_\alpha : \alpha \in A\}$ be a family of sets; the *nerve* $N(\mathcal{K})$ of $\mathcal{K}$ is an abstract simplicial complex on A defined as follows:

$$N(\mathcal{K}) = \left\{ B \subset A : \bigcap_{\alpha \in B} K_\alpha \neq \varnothing \right\}. \tag{3.1}$$

DEFINITION. Let C be a simplicial complex and let S be a face of C of dimension $< d$, which is included in a unique maximal face of C (such a face S is called a *free face*). The operation of deleting S and every face that includes S in called an *elementary d-collapse.* A finite sequence of elementary d-collapses is called a *d-collapse.* C is *d-collapsible* if C d-collapses to the void complex.

DEFINITION. A simplical complex C is *d-representable* if C is the nerve of a finite family of convex sets in $\mathbf{R}^d$.

We shall use the following theorem:

THEOREM (Wegner [14]). *Every d-representable complex is d-collapsible.*

For $n > r > 0$ put $N = \{1, 2, \ldots, n\}$, $R = \{1, 2, \ldots, r\}$, $\bar{R} = N \setminus R$, and define:

$$P_k(n, d, r) = \{S \in N^{(k)} : |S \cap \bar{R}| \leqq d\}, \tag{3.2}$$

$$p_k(n, d, r) = |P_k(n, d, r)| = \sum_{i=0}^{d} \binom{r}{k-i}\binom{n-r}{i}. \tag{3.3}$$

Note that $p_k(n, d, r) = \binom{n}{k}$ for $k \leqq d$, $p_{d+1}(n, d, r) = \binom{n}{d+1} - \binom{n-r}{d+1}$ and $p_{d+r}(n, d, r) = \binom{n-r}{d}$.

For a simplicial complex C, $f_k(C) = |\{S \in C : \dim S = k\}|$.

Note that if $\mathcal{K} = \mathcal{K}(n, d, r) = \{K_1, \ldots, K_n\}$ where $n \geqq d + r$, $K_1 = \cdots = K_r = R^d$ and $K_{r+1}, \ldots, K_n$ are hyperplanes in general position in R^d, then $\dim N(\mathcal{K}) = d + r - 1$ and $N(\mathcal{K}) = \{S \subset N : |S \cap \bar{R}| \leqq d\}$ and thus $f_{k-1}(N(\mathcal{K})) = p_k(n, d, r)$ for all k.

In view of Wegner's Theorem, the conjecture of Katchalski and Perles follows from the following theorem:

THEOREM 3.1. *Let C be a d-collapsible simplicial complex of dimension $\leqq d + r - 1$ $(r > 0)$, with n vertices. Then*

$$f_{k-1}(C) \leqq p_k(n, d, r) \qquad \text{for all } k \geqq 0.$$

In the proof of Theorem 3.1 we shall need the following lemma:

LEMMA 3.2. *Every elementary d-collapse can be replaced by a finite sequence of special elementary d-collapses of the following types:*

(A) *Removal of a maximal face of dimension $< d$.*

(B) *Removal of a free face S of dimension $d - 1$ and all the faces that include S.*

PROOF. Suppose S is a free face of C, M is the unique maximal face of C that includes S and $C^* = C \setminus \{F : F \supset S\}$.

If $\dim M \leqq d-1$ then we can pass from C to C^* by removing all the faces that include S one by one.

If $\dim S = d-1$ there is nothing to do.

If $\dim M > d-1 > \dim S$, proceed by induction on the difference $\dim M - \dim S$. Choose a vertex $q \in M \setminus S$ and put $S' = S \cup \{q\}$, $M' = M \setminus \{q\}$, $C' = C \setminus \{F : F \supset S'\}$. Then S' is a free face of C, S is a free face of C', the unique maximal face of C' that includes S is M', and $C^* = C' \setminus \{F : F \supset S\}$. The lemma holds for the elementary d-collapses $C \to C'$ and $C' \to C^*$ either by the induction hypothesis or by the trivial cases that were settled before. Therefore it holds also for the d-collapse $C \to C^*$.

PROOF OF THEOREM 3.1. There is nothing to prove unless $d < k \leqq d + r$. Suppose $\operatorname{vert} C = N$. Define (with the notation of section 2)

$$W_k = \operatorname{span}(e_T : T \in C, |T| = k) \subset \wedge^k V \qquad (\dim W_k = f_{k-1}(C)),$$

$$A_k = \{m \in \wedge^k V : \forall (T \in R^{(k-d)})\ f_T \llcorner m = 0\}.$$

The following two assertions clearly imply Theorem 3.1:

(1) $\dim A_k \geqq \binom{n}{k} - p_k(n,d,r)$.

(2) $A_k \cap W_k = \{0\}$.

(1) follows from the fact that if $S \in N^{(k)}$ and $S \notin P_k(n,d,r)$ then $|S \cap R| < k-d$, and thus $T \in R^{(k-d)}$ implies $S \not\supset T$ and therefore $f_T \llcorner f_S = 0$ (see (2.2)), i.e., $f_S \in A_k$.

For the proof of (2), assume that $0 \neq m \in A_k \cap W_k$, $m = \Sigma\{\alpha_S e_S : S \in C, |S| = k\}$. Choose a fixed sequence of special elementary d-collapses (as in Lemma 3.2) that reduces C to the void complex $\varnothing$, and consider the first step $C_{i-1} \to C_i$ in which a face U ($|U| = k$) with nonzero coefficient α_U is deleted.

Denote by L and M the free face and the maximal face in this step. Clearly $L \subset U \subset M$, $|M| \geqq |U| = k > d$, and therefore $|L| = d$. (See Lemma 3.2.)

If $T \in R^{(k-d)}$ and $S \in N^{(k)}$ then by (2.1)

$$f_T = \sum\{(\det A_{T|P})e_P : P \in N^{(k-d)}\}$$

and therefore

$$\langle e_L, f_T \llcorner e_S\rangle = \langle e_L \wedge f_T, e_S\rangle = \sum\{\det A_{T|P} < e_L \wedge e_P, e_S > : P \in N^{(k-d)}\}.$$

If $S \not\supset L$ then $\langle e_L \wedge e_P, e_S\rangle = 0$ for all $P \in N^{(k-d)}$ and therefore $\langle e_L, f_T \llcorner e_S\rangle = 0$.

If $S \supset L$ then $\langle e_L \wedge e_P, e_S \rangle = 0$ unless $P = S \setminus L$, and therefore

$$\langle e_L, f_T \llcorner e_S \rangle = \langle e_L \wedge e_{S \setminus L}, e_S \rangle \cdot \det A_{T|S \setminus L}.$$

Here $\langle e_L \wedge e_{S\setminus L}, e_S \rangle$ equals ± 1 (and depends only on L and S). From the assumption that $m \in A_k$ it follows that for all $T \in R^{(k-d)}$

$$0 = \langle e_L, f_T \llcorner m \rangle = \sum_{\substack{S \in C \\ |S| = k}} \alpha_S \langle e_L, f_T \llcorner e_S \rangle$$

(3.3)
$$= \sum_{\substack{S \in C_{i-1} \\ |S| = k}} \alpha_S \langle e_L, f_T \llcorner e_S \rangle = \sum_{\substack{M \supset S \supset L \\ |S| = k}} \alpha_S \langle e_L, f_T \llcorner e_S \rangle$$

$$= \sum_{\substack{M \supset S \supset L \\ |S| = k}} \alpha_S \langle e_L \wedge e_{S \setminus L}, e_S \rangle \det A_{T|S \setminus L}.$$

Since $M \supset U \supset L$, $|U| = k$ and $e_U \neq 0$, (3.3) is a linear dependence of the columns of $C_{k-d}(A_{R|M \setminus L})$; but since by our asumption on the basis $f_1, \ldots, f_n$ the columns of $A_{R|M \setminus L}$ are linearly independent, this is impossible. (See section 2). ■

4. Applications

Our first application (Theorem 4.3) is a sharp and detailed version of a "fractional" Helly type theorem of Katchalski and Liu [7].

DEFINITION 4.1. For $0 \leqq \rho \leqq 1$ and $1 \leqq d < k \leqq n$, let $\alpha = \alpha(\rho, d, k, n)$ be the smallest real number with the following property:

(P) If $\mathcal{K}$ is a family of n convex sets in R^d, and if the number of intersecting k-subfamilies of $\mathcal{K}$ is $> \alpha\binom{n}{k}$ (i.e., $f_{k-1}(N(\mathcal{K})) > \alpha\binom{n}{k}$), then $\mathcal{K}$ has an intersecting subfamily of size $> \rho n$ (i.e., $1 + \dim N(\mathcal{K}) > \rho n$).

REMARKS. (1) A family of n-hyperplanes in general position in R^k has $\binom{n}{k}$ intersecting subfamilies of size k and none of size $> k$. Therefore there is no point in defining $\alpha(\rho, d, k, n)$ for $k \leqq d$.

(2) From Definition 4.1 it follows directly that $\alpha(\rho, d, k, n)$ is non-decreasing in ρ (for fixed d, k, n) with $\alpha(\rho, d, k, n) = 0$ if $\rho n < k$ and $\alpha(1, d, k, n) = 1$. $\alpha(\rho, d, k, n)$ is also non-decreasing in d (for fixed ρ, k, n).

(3) A simple incidence counting argument shows that for every simplicial complex C with n vertices

(4.1)
$$(k+1) f_k(C) \leqq (n-k) f_{k-1}(C)$$

and therefore the ratio $\binom{n}{k}^{-1} f_{k-1}(C)$ is a non-decreasing function of k for $0 \leqq k \leqq n$. It follows that $\alpha(\rho, d, k, n)$ is non-increasing in k (for fixed ρ, d, n).

From Definition 4.1 one can easily derive by contraposition the following, more explicit, expression for $\alpha(\rho, d, k, n)$:

$$\alpha(\rho, d, k, n) = \max\{\tbinom{n}{k}^{-1} f_{k-1}(N(\mathcal{K})) : \mathcal{K} \text{ is a family of } n \text{ convex sets in } R^d \text{ and } 1 + \dim N(\mathcal{K}) \leqq \rho n\}. \tag{4.2}$$

From Theorem 3.1 it follows that if $\rho n \geqq d + 1$

$$\alpha(\rho, d, k, n) = \max\{\tbinom{n}{k}^{-1} p_k(n, d, r) : d + 1 \leqq d + r \leqq \rho n\}. \tag{4.3}$$

Since $p_k(n, d, r)$ is a non-decreasing function of r (for fixed k, n and d) we obtain

$$\begin{aligned} \alpha(\rho, d, k, n) &= \tbinom{n}{k}^{-1} p_k(n, d, [\rho n] - d) \\ &= \tbinom{n}{k}^{-1} \sum_{i=0}^{d} \tbinom{[\rho n] - d}{k - i} \tbinom{n - [\rho n] + d}{i} \\ &= \frac{1}{(n)_k} \sum_{i=0}^{d} \tbinom{k}{i} ([\rho n] - d)_{k-i} (n - [\rho n] + d)_i. \end{aligned} \tag{4.4}$$

(Here $(n)_k$ stand for the factorial $\prod_{i=0}^{k-1} (n - i)$.)

From the last equation one can easily obtain the limit

$$\lim_{n \to \infty} \alpha(\rho, d, k, n) = \sum_{i=0}^{d} \tbinom{k}{i} \rho^{k-i} (1 - \rho)^i. \tag{4.5}$$

DEFINITION 4.2. For $0 < \rho < 1$ and $d \geqq 1$

$$\alpha(\rho, d, k) = \sup_{n \geq k} \alpha(\rho, d, k, n), \tag{4.6}$$

$$\alpha(\rho, d) = \sup_{k > d} \alpha(\rho, d, k). \tag{4.7}$$

$\alpha(\rho, d, k)$ is the smallest real number with property P for *all* values of $n \geqq k$. In other words if $\mathcal{K}$ is any finite family of convex sets in R^d with $|\mathcal{K}| \geqq k$, and if the number of intersecting subfamilies of $\mathcal{K}$ is $> \alpha\binom{n}{k}$, where $n = |\mathcal{K}|$, then $\mathcal{K}$ has an intersecting subfamily of size $> \rho n$.

From remark (3) after Definition 4.1 it follows that

$$\alpha(\rho, d) = \alpha(\rho, d, d + 1). \tag{4.8}$$

A priori it is not clear that $\alpha(\rho,d,k)<1$ for some $k>d$. Katchalski and Liu proved in [7], among other results, that:

$$\text{for any } d>0 \text{ and } \rho<1, \qquad \alpha(\rho,d)<1.$$

They also proved that $\lim_{\rho\to 0}\alpha(\rho,d)=0$.

The exact value of $\alpha(\rho,d)$ was unknown for $d>1$. For $d=1$ Abbot and Katchalski [1] proved that $\alpha(\rho,1)=1-(1-\rho)^2$.

THEOREM 4.3. *For* $0<\rho<1$, $0<d<k$

$$\alpha(\rho,d,k)=\sum_{i=0}^{d}\binom{k}{i}\rho^{k-i}(1-\rho)^i=1-\sum_{i=d+1}^{n}\binom{k}{i}\rho^{k-i}(1-\rho)^i. \tag{4.10}$$

In particular $\alpha(\rho,d)=\alpha(\rho,d,d+1)=1-(1-\rho)^{d+1}$.

PROOF. In view of formula (4.5) it clearly suffices to show that

$$\alpha(\rho,d,k,n)\leqq\lim_{n\to\infty}\alpha(\rho,d,k,n)\qquad\text{for all } n\geqq k.$$

We shall prove this by showing that

$$\alpha(\rho,d,k,2n)\geqq\alpha(\rho,d,k,n)\qquad\text{for all } n\geqq k. \tag{4.11}$$

If $\rho n\leqq d$ then $\alpha(\rho,d,k,n)=0$ and (4.11) holds automatically.

Now suppose $n\geqq k>d$, $\rho n>d$. Let $\mathcal{K}$ be the family of n convex sets in R^d (defined in section 3) such that $f_{k-1}(N(\mathcal{K}))=p_k(n,d,[\rho n]-d)$. Define $2\mathcal{K}$ to be the family of $2n$ convex sets in R^d consisting of two copies of each member of $\mathcal{K}$. Obviously $1+\dim N(\mathcal{K})=2[\rho n]\leqq\rho 2n$. It is easy to see that

$$f_{k-1}(N(2\mathcal{K}))=\sum_{i=0}^{[k/2]}f_{k-1-i}(N(\mathcal{K}))\binom{k-i}{i}2^{k-2i} \tag{4.12}$$

and similarly

$$\binom{2n}{k}=\sum_{i=0}^{[k/2]}\binom{n}{k-i}\binom{k-i}{i}2^{k-2i}. \tag{4.13}$$

Now for $0\leqq i\leqq[k/2]$,

$$\begin{aligned}2^{k-2i}\binom{k-i}{i}f_{k-1-i}(N(\mathcal{K}))/2^{k-2i}\binom{k-i}{i}\binom{n}{k-i}&=\binom{n}{k-i}^{-1}f_{k-1-i}(N(\mathcal{K}))\\&\geqq\binom{n}{k}^{-1}f_{k-1}(N(\mathcal{K}))\end{aligned} \tag{4.14}$$

(see remark (3) after Definition 4.1). Therefore

$$\alpha(\rho,d,k,2n)\geqq\binom{2n}{k}^{-1}f_{k-1}(N(2\mathcal{K}))\geqq\binom{n}{k}^{-1}f_{k-1}(N(\mathcal{K}))=\alpha(\rho,d,k,n). \tag{4.15}$$

REMARK. For $k = d+1$ one can show directly that

$$\alpha(\rho, d, d+1, n) = \binom{n}{d+1}^{-1}\left(\binom{n}{d+1} - \binom{[\rho n]}{d+1}\right) \leqq 1 - (1-\rho)^{d+1}.$$

However, for $k > d+1$, we do not have a direct agrument to prove Theorem 4.3, and we had to invoke again Theorem 3.1.

From Theorem 4.6 it follows that for any fixed $d \geqq 1$ and $0 < \rho < 1$

$$\lim_{k \to \infty} \sup_{n > k} \alpha(\rho, d, k, n) = \lim_{k \to \infty} \alpha(\rho, d, k) = 0. \tag{4.16}$$

In words:

THEOREM 4.4. *Let $d \geqq 1$ and $0 < \rho < 1$ be fixed. Then for each $\varepsilon > 0$ there is a k such that every family of n convex sets in R^d with more than $\varepsilon\binom{n}{k}$ intersecting subfamilies of size k has an intersecting subfamily of size $> \rho n$.*

Furthermore we can show:

THEOREM 4.5. *Let $d \geqq 1$ and $0 < c < 1$ be fixed. Then for each $0 < \varepsilon < 1$ there is a natural number $s = s(d, c, \varepsilon)$ such that every family of n convex sets in R^d with more than $\varepsilon\binom{n}{[cn]}$ intersecting subfamilies of size $[cn]$ has an intersecting subfamily of size $> n - s$.*

PROOF. The theorem is equivalent to the assertion:

$$\lim_{s \to \infty} \sup_{n > s} \alpha(1 - s/n, d, [cn], n) = 0$$

Assume $n > s > 2c^{-1}(d+1)$. (This implies $cn - d - 1 \geqq \frac{1}{2}cn$.)

$$\begin{aligned}
\alpha(1 - s/n, d, [cn], n) \leqq \alpha(1 - s/n, d, [cn]) &= \sum_{i=0}^{d} \binom{[cn]}{i} \left(\frac{n-s}{n}\right)^{[cn]-i} \left(\frac{s}{n}\right)^{i} \\
&\leqq \sum_{i=0}^{d} \frac{(cn)^i}{i!} \left(1 - \frac{s}{n}\right)^{[cn]-i} \left(\frac{s}{n}\right)^{i} \\
&= \sum_{i=0}^{d} \frac{(cs)^i}{i!} \left(1 - \frac{s}{n}\right)^{[cn]-i} \\
&\leqq \sum_{i=0}^{d} \frac{(cs)^i}{i!} \left(1 - \frac{s}{n}\right)^{cn-d-1} \\
&\leqq \left(1 - \frac{s}{n}\right)^{cn/2} \sum_{i=0}^{d} \frac{(cs)^i}{i!} \\
&\leqq e^{-cs/2} \sum_{i=0}^{d} \frac{(cs)^i}{i!} \\
&\xrightarrow{s \to \infty} 0.
\end{aligned}$$

Let $s(d, c, \varepsilon)$ be the smallest natural number satisfying the assertion of Theorem 4.5. Using somewhat more delicate estimates we can prove that

$$s(d, c, \varepsilon) \leqq c^{-1}[(2d + \log \varepsilon^{-1}) + (\log(2d + \log \varepsilon^{-1}))^2].$$

Finally we would like to draw attention to the following special case of Theorem 3.1.

THEOREM 4.6. *Let $d \geqq 1$, $k \geqq 0$ be fixed. Suppose $\mathcal{K}$ is a family of convex sets in R^d, and suppose $\mathcal{K}$ has an intersecting subfamily that misses at most k members of $\mathcal{K}$. Then the number of intersecting subfamilies of $\mathcal{K}$ of maximal size is at most* $\binom{k+d}{d}$.

(Note that this bound is independent of the size of $\mathcal{K}$.)

5. Applications to extremal set theory

The condition of d-collapsibility in Theorem 3.1 can be weakened.

DEFINITION 5.1. A simplicial complex C is *weakly (d, r)-collapsible*, if C can be reduced to the void complex $\varnothing$ by a sequence of steps of the following two types:

(A′) Removal of a maximal face of dimension less than $d-1$.

(B′) Removal of a "weakly free" face S of dimension $\leqq d-1$ and all the faces that include S, where "S weakly free" means

$$|\cup\{T \in C : T \supset S\}| \leqq d + r.$$

(Note that if C is weakly (d, r)-collapsible, then $\dim C \leqq d + r - 1$.)

THEOREM 5.2. *Let C be a weakly (d, r)-collapsible complex, with n vertices. Then*

$$f_{k-1}(C) \leqq p_k(n, d, r) \qquad \textit{for all } k \geqq 0.$$

This strengthened version of Theorem 3.1 requires almost no change in the proof of Lemma 3.2 and Theorem 3.1. The only essential difference is the replacement of the unique maximal face M that includes the free face L by the union of all faces that include L.

Theorem 5.2 can be rephrased as an extremal set theoretic result as follows:

Let $\mathcal{B}$ be a family of k-subsets of an n-set N. For $L \in N^{(d)}$, $d < k$, define

$$\deg_{\mathcal{B}} L = |\{C \in \mathcal{B} : L \subset C\}|. \tag{5.1}$$

THEOREM 5.3. *Suppose $1 \leqq d < k \leqq d + r < n$. If $\mathscr{C}$ is a family of k-subsets of an n-set N and $|\mathscr{C}| > p_k(n, d, r)$, then there exists a nonempty subfamily $\mathscr{B}$ of $\mathscr{C}$ such that for all $L \in N^{(d)}$ either $\deg_{\mathscr{B}} L = 0$ or $|\cup\{C \in \mathscr{B} : L \subset C\}| > d + r$.*

One easily verifies that the conclusion of Theorem 5.1 fails for the family $P_k(n, d, r)$ defined in the previous section. This shows that Theorem 5.3 is best possible for all admissible values of the parameters k, n, d, r. Two special cases of Theorem 5.3 are of particular interest:

THEOREM 5.4. $(d = k - 1)$ *If $\mathscr{C}$ is a family of k-subsets of an n-set N and $|\mathscr{C}| > \binom{n}{k} - \binom{n-r}{k}$, then there is a subfamily $\mathscr{B}$ of $\mathscr{C}$ such that for every $L \in N^{(k-1)}$, either $\deg_{\mathscr{B}} L = 0$ or $\deg_{\mathscr{B}} L > r$.*

THEOREM 5.5.[†] $(r = k - d)$ *If $\mathscr{C}$ is a family of k-subsets of an n-set N and $|\mathscr{C}| > \binom{n-k+d}{d}$, then there is a subfamily $\mathscr{B}$ of $\mathscr{C}$ such that for every $L \in N^{(d)}$, $\deg_{\mathscr{B}} L \neq 1$.*

6. Remarks

1. The Katchalski–Perles conjecture was proved independently by J. Eckhoff (to appear). His proof is more elementary than ours and is based upon the following definitions and observations.

(I) Let C be a d-representable complex on n-vertices. The h-vector of C, $h(C) = (h_0, h_1, \ldots)$, is defined by

$$h_k = \begin{cases} f_k(C) & k = 0, 1, \ldots, d-1, \\ \sum_{j \geq 0} (-1)^j \binom{k+j-d}{j} f_{k+j}(C) & k = d, d+1, \ldots. \end{cases} \tag{6.1}$$

Eckhoff noticed that the Katchalski–Perles conjecture is equivalent to the inequality

$$h_k \leqq \binom{n-k+d-1}{d} \qquad \text{for all } k \geqq d. \tag{6.2}$$

(II) The effect of an elementary d-collapse on the h-vector of C is rather simple and was computed by Eckhoff in [3].

(III) Eckhoff proved that a d-representable complex is *strongly d-collapsible*, where strongly d-collapsible roughly means that the faces of the complex can be d-collapsed in such a way that all the faces that contain a given vertex are left to the end of the collapse process. This fact enables him to use (II) and to give an inductive proof of (6.2).

[†] *Note added in proof.* Theorem 5.5 was proved independently by P. Frankl [15].

Notes

(a) Eckhoff's proof resembles McMullen's proof of the upper bound conjecture for convex polytopes [13].†

(b) Theorem 3.1 and the results of section 5 do not follow from Eckhoff's proof.

(c) The reader should note that d-collapsibility and even strong d-collapsibility are far from being characterizations of d-representable complexes.

2. After finding upper bounds for the f-vectors of d-representable complexes, the next step would be to give a full characterization of f-vectors of such complexes. Indeed in 1973 (see [3]) Eckhoff introduced a system of inequalities for the h-vector (see above) of a simplicial complex C, and conjectured that these inequalities characterize h-vectors of d-representable complexes. Using a refinement of the methods employed in the proof of Theorem 3.1, we can show that h-vectors of d-representable complexes do indeed satisfy Eckhoff's inequalities. This will appear in [5].

3. For a complex C and a face $S \in C$, the quotient complex C/S (known also as the link of S in C) is defined by

$$C/S = \{T \setminus S : T \in C,\ T \supset S\}. \tag{6.1}$$

It follows from a theorem of Leray [9] that if $\mathcal{K}$ is a finite family of homology cells in R^d (or in any d-dimensional manifold with boundary) such that every non-empty intersection of members of $\mathcal{K}$ is again a homology cell, and if $C = N(\mathcal{K})$, then

$$H_l(C/S) = 0 \qquad \text{for every } l \geqq d \text{ and every } S \in C. \tag{6.2}$$

It seems that condition (6.2), which is obviously weaker than d-collapsibility, is sufficient to imply the upper bounds of Theorem 3.1, but we can prove it only for $r = 1, 2, 3$ (i.e., if $\dim C \leqq d + 2$).††

4. It would be interesting of find analogs of our main theorem for families of sets belonging to a restricted class of convex sets in R^d, such as the class $\mathcal{P}_d$ of d-dimensional parallelotopes with edges parallel to the coordinate axes in R^d, or the classes TC of translates of a fixed convex set C in R^d. We shall make a conjecture for the first class:

† *Note added in proof.* For a proof of McMullen's upper bound theorem based on the approach of this paper, see [16]. (There we provide also a simpler proof for Theorem 3.1.)

†† *Note added in proof.* This has now been proved for every $r > 0$.

CONJECTURE 6.1. *Let $\mathscr{C}=\{P_1,\ldots,P_n\}$ be a family of n d-dimensional parallelotopes with edges parallel to the coordinate axes. Suppose that every $(r+1)$-subfamily of $\mathscr{C}$ has empty intersection. Then $f_1(N(\mathscr{C}))\leqq g(n,d,r)$, where*

$$g(n,d,r)=t_r(n) \qquad \text{for } r\leqq d,$$

$$g(n,d,r)=t_d(n-r+d)+(n-r+d)(r-d)+\binom{r-d}{2} \qquad \text{for } r\geqq d.$$

Here $t_r(d)$ (t — after Turán) is the maximal possible number of edges in a graph on n vertices with no complete subgraph on $r+1$ vertices.

We shall now describe a family $\mathscr{C}=\mathscr{C}(n,d,r)$ $(n\geq r)$ which satisfies $\dim(N(\mathscr{C}))<r$ and $f_1(N(\mathscr{C}))=g(n,d,r)$. For $1\leqq i\leqq d$ put $H_i=\{x\in R^d : x_i=0\}$.

If $r\leqq d$, partition the number n into r almost equal parts $n_1,\ldots,n_r$.

$$\left[\frac{n}{r}\right]\leqq n_i\leqq\left\lceil\frac{n}{r}\right\rceil \qquad \text{for } 1\leqq i\leqq r.$$

$\mathscr{C}$ consists of n_i distinct translates of H_i, for $1\leqq i\leqq r$.

If $r>d$, partition the number $n-(r-d)$ into d almost equal parts $n_1,\ldots,n_d$. $\mathscr{C}$ consists of $r-d$ copies of R^d, plus n_i distinct translates of H_i for $1\leqq i\leqq d$.

Analogs of Theorem 4.3 for the class $\mathscr{P}_d$ were found by Katchalski in [8].

5. The following result of N. Linial and B. R. Rothchild [10] resembles quite closely our Theorem 4.3. Suppose $0<l<k<n$. If $\mathscr{C}$ is a family of k-subsets of an n-set N and $|\mathscr{C}|>f(n,k,l)$ (where f is the function defined below) then there is a subfamily $\mathscr{B}$ of $\mathscr{C}$ such that for every $L\in N^{(l)}$, $\deg_{\mathscr{B}}L$ is even.

DEFINITION. $f(n,k,l)=\sum_{r=1}^{l}c(r)\binom{n}{l-r}$, where the coefficients $c(r)$ are defined as follows: Put $h=k-l$ and suppose $h=\sum\{2^i : i\in H\}$ with $\min H=h_1$. For $0<r\leqq l$, suppose $r=\sum\{2^i : i\in R\}$ with $r_1=\min R$. Now

$$c(r)=\begin{cases}-1 & \text{if } r_1=h_1 \text{ and } R\cap H=\{h_1\},\\ 1 & \text{if } r_1>h_1 \text{ and } R\cap H=\varnothing,\\ 0 & \text{otherwise.}\end{cases}$$

(Our definition of $f(n,k,l)$ is a simplification of the one given in [10].)

6. The ideas used here to prove Theorem 3.1 can be used to define a new concept of connectivity for graphs and to solve some extremal problems in graph theory. This will be done in [6].

7. The correspondence between families of sets and subspaces of the exterior algebra is also used by Ofer Gabber and the author in [4] to find and prove geometric analogs for extremal set theoretic results. A similar approach has been taken by Lovász in [11].

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